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De iterata integratione formularum integralium, dum aliquis exponens pro variabili assumitur

Leonhard Euler

DE ITERATA INTEGRATIONE FORMVLARVM INTEGRALIVM DVM ALIQVIS EXPONENS PRO VARIABILI ASSVMITVR.

Auctore

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Problema 1.

Cum sit $\int x^{\theta-1}\partial x \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{1}{\theta}$, *hanc formulam denuo integrare, sumto exponente* θ *variabili.*

Solutio.

§. 1. Quoniam hic de integratione agitur, ut ea determinetur, integrale ita capi assumamus, ut evanescat certo casu, posito scilicet $\theta = \alpha$. Multiplicetur ergo utrinque per elementum $\partial\theta$, et integratione juxta hanc legem instituta pro parte dextra habebimus $\int\frac{\partial\theta}{\theta} = l\,\theta - l\,\alpha = l\,\frac{\theta}{\alpha}$. At pro parte sinistra notum est, hanc integrationem a signo summatorio $\int$ penitus non turbari, et quia jam sola littera θ pro variabili habetur, $\frac{\partial x}{x}$ vero ut constans spectatur, ob $x^{\theta-1}\partial x = \frac{\partial x}{x}x^{\theta}$, habebimus

$\int$

$$\int x^{\theta} \partial \theta = \frac{x^{\theta}}{l\,x} - \frac{x^{\alpha}}{l\,x};$$

quo valore ſubſtituto membrum ſiniſtrum erit

$$\int \frac{\partial x}{x} \cdot \frac{x^{\theta} - x^{\alpha}}{l\,x},$$

quamobrem iſta integratio iterata nos perducit ad hanc aequationem:

$$\int \frac{x^{\theta-1} - x^{\alpha-1}\,\partial x}{l\,x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l\frac{\theta}{\alpha}.$$

Corollarium 1.

§. 2. Si eodem modo formula integralis

$$\int x^{n+\theta-1}\,\partial x \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{1}{n+\theta}$$

denuo integretur, ſumto θ variabili, reperietur haec aequatio integrata:

$$\int (x^{n+\theta-1} - x^{n+\alpha-1})\frac{\partial x}{l\,x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l\frac{n+\theta}{n+\alpha}.$$

At ſi θ negatiue capiatur, tum etiam α negatiue accipi debebit, vnde aequatio denuo integrata haec prodibit:

$$\int (x^{n-\theta-1} - x^{n-\alpha-1})\frac{\partial x}{l\,x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l\frac{n-\theta}{n-\alpha}.$$

Corollarium 2.

§. 3. Hic igitur notentur iſtae integrationes, quas in parte ſiniſtra inſtitui oportet, et quibus pro aliis formulis in poſterum erit vtendum, vbi ſemper aſſumamus, integralia ita capi debere, vt euaneſcant poſito $\theta = \alpha$. Primo ſcilicet erit

$$\int x^{\theta}\,\partial\,\theta = \frac{x^{\theta} - x^{\alpha}}{l\,x}.$$

Praeterea vero ſimili modo

$$\int x^{n+\theta}\,\partial\,\theta = \frac{x^{n+\theta} - x^{n+\alpha}}{l\,x};$$

atque hinc porro intelligitur fore

$$\int x^{n+\lambda\theta}\,\partial\,\theta = \frac{x^{n+\lambda\theta} - x^{n+\lambda\alpha}}{\lambda\,l\,x},$$

vnde patet, ſi λ capiatur negative, fore

$$\int x^{n-\lambda\theta}\,\partial\,\theta = \frac{x^{n-\lambda\theta} - x^{n-\lambda\alpha}}{-\lambda\,l\,x}.$$

Problema 2.

Cum ſit, vti jam ſaepius eſt oſtenſum,

$$\int \frac{x^{\theta-1}\,\partial\,x}{1+x^{\nu}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi}{\nu\,\text{ſin.}\,\frac{\theta\pi}{\nu}},$$

hanc aequationem denuo integrare, ſumto exponente θ pro variabili.

Solutio.

§. 4. Perpetuo hic, vt hactenus, integralia ita accipi ſtatuamus, vt evaneſcant poſito $\theta = \alpha$; quo obſervato pro parte dextra habebimus $\int \frac{\pi\,\partial\,\theta}{\nu\,\text{ſin.}\,\frac{\theta\pi}{\nu}}$, quae formula poſito $\frac{\theta\pi}{\nu} = \Phi$ abit in hanc: $\int \frac{\partial\Phi}{\text{ſin.}\,\Phi}$, cujus integrale novimus eſſe $l\,\text{tang.}\,\frac{1}{2}\Phi$; quamobrem adjecta debita conſtante pro hac parte habebimus

$$\int \frac{\pi\,\partial\theta}{\nu\,\text{ſin.}\,\frac{\theta\pi}{\nu}} = l\,\text{tang.}\,\frac{\theta\pi}{2\nu} - l\,\text{tang.}\,\frac{\alpha\pi}{2\nu} = l\,\frac{\text{tang.}\,\frac{\theta\pi}{2\nu}}{\text{tang.}\,\frac{\alpha\pi}{2\nu}}.$$

Pro

Pro parte autem finiſtra, vbi ſolus factor $x^{\theta-1}$ eſt variabilis, erit

$$\int x^{\theta-1}\partial\theta = \frac{x^{\theta-1} - x^{\alpha-1}}{l\,x}.$$

Hoc igitur valore introducto formula noſtra integralis denuo integrata erit

$$\int \frac{\partial x\,(x^{\theta-1} - x^{\alpha-1})}{(1+x^{\nu})\,l\,x}\left[\begin{array}{l}\text{ab } x=0\\ \text{ad } x=\infty\end{array}\right] = l\,\frac{\text{tang.}\,\frac{\theta\pi}{2\nu}}{\text{tang.}\,\frac{\alpha\pi}{2\nu}}.$$

Corollarium.

§. 5. Quodſi ergo ſumamus $\alpha = \frac{1}{2}\nu$, quoniam tang. $\frac{\pi}{4} = 1$, hoc caſu, ponendo potius $\nu = 2\alpha$, habebimus hanc aequationem integralem ſatis memorabilem:

$$\int \frac{\partial x\,(x^{\theta-1} - x^{\alpha-1})}{(1+x^{2\alpha})\,l\,x}\left[\begin{array}{l}\text{ab } x=0\\ \text{ad } x=\infty\end{array}\right] = l\,\text{tang.}\,\frac{\theta\pi}{4\alpha}.$$

Problema 3.

Cum ſit, vti jam ſatis conſtat:

$$\int \frac{(x^{\theta-1} + x^{\nu-\theta-1})\,\partial x}{1+x^{\nu}}\left[\begin{array}{l}\text{ab } x=0\\ \text{ad } x=1\end{array}\right] = \frac{\pi}{\nu\,\text{ſin.}\,\frac{\theta\pi}{\nu}},$$

hanc aequationem denuo integrare per exponentem variabilem θ, ita vt integralia evaneſcant poſito $\theta = \alpha$.

Solutio.

§. 6. Multiplicando igitur per $\partial\theta$ et integrando, pro parte dextra, prorſus vt in praecedente problemate, habebimus

$$l\,\frac{\text{tang.}\,\frac{\theta\pi}{2\nu}}{\text{tang.}\,\frac{\alpha\pi}{2\nu}}.$$

Pro

Pro parte autem ſiniſtra, quia formula $\frac{\partial x}{1+x^\nu}$ eſt conſtans, et exponens θ in duobus terminis occurrit, pro priore termino habebimus

$$\int x^{\theta-1}\partial\theta = \frac{x^{\theta-1}-x^{\alpha-1}}{l\,x},$$

pro altero vero termino ex §. 3. habebimus

$$\int x^{\nu-\theta-1}\partial\theta = \frac{x^{\nu-\alpha-1}-x^{\nu-\theta-1}}{l\,x},$$

quibus valoribus ſubſtitutis orietur iſta noua integratio:

$$\int\frac{\partial x}{l\,x}\cdot\frac{x^{\theta-1}-x^{\alpha-1}+x^{\nu-\alpha-1}-x^{\nu-\theta-1}}{1+x^\nu}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\frac{\text{tang.}\frac{\theta\pi}{2\nu}}{\text{tang.}\frac{\alpha\pi}{2\nu}}.$$

Corollarium 1.

§. 7. Iſta aequatio aliquanto ſuccinctius ita repraeſentari poteſt:

$$\int\frac{\partial x}{x\,l\,x}\frac{(x^\theta-x^\alpha+x^{\nu-\alpha}-x^{\nu-\theta})}{1+x^\nu}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\frac{\text{tang.}\frac{\theta\pi}{2\nu}}{\text{tang.}\frac{\alpha\pi}{2\nu}}$$

vbi cum ſit $x^{\nu-\alpha}-x^{\nu-\theta}=x^{\nu-\alpha-\theta}(x^\theta-x^\alpha)$, iſta aequatio ita commodius per factores repraeſentari poterit:

$$\int\frac{\partial x}{x\,l\,x}\frac{(x^\theta-x^\alpha)(1+x^{\nu-\alpha-\theta})}{1+x^\nu}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} = l\frac{\text{tang.}\frac{\theta\pi}{2\nu}}{\text{tang.}\frac{\alpha\pi}{2\nu}}.$$

Corollarium 2.

§. 8. Quodſi hic capiamus $\theta=\nu-\alpha$, vt fiat $x^{\nu-\alpha-\theta}=1$, pro parte dextra erit tang. $\frac{(\nu-\alpha)\pi}{2\nu}=$ cotang. $\frac{\alpha\pi}{2\nu}$, vnde totum hoc membrum erit $2\,l$ cot. $\frac{\alpha\pi}{2\nu}$; quare cum pro parte ſiniſtra

factor

factor $1+x^{\nu-\alpha-\theta}$ evadat $=2$, vtrinque per 2 dividendo habebimus

$$\int \frac{\partial x}{x\, l\, x} \cdot \frac{x^{\nu-\alpha}-x^{\alpha}}{1+x^{\nu}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = l \cot. \frac{\alpha\pi}{2\nu}.$$

Corollarium 3.

§. 9. Quodsi sumamus $\nu=2\alpha$, vt fiat tang. $\frac{\alpha\pi}{2\nu}=1$, pro parte sinistra factor $1+x^{\nu-\alpha-\theta}$ abit in $1+x^{\alpha-\theta}$, dum prior factor $x^{\theta}-x^{\alpha}$ ita repraesentari potest: $x^{\theta}(1-x^{\alpha-\theta})$; vnde amborum productum erit $x^{\theta}(1-x^{2\alpha-2\theta})$, quamobrem integratio nostra ita se habebit:

$$\int \frac{x^{\theta-1}\partial x}{l\, x} \cdot \frac{1-x^{2\alpha-2\theta}}{1+x^{2\alpha}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = l \text{ tang. } \frac{\theta\pi}{4\alpha}.$$

Scholion.

§. 10. Istae integrationes eo majorem attentionem merentur, quod in iis tres exponentes α, θ, ν indefiniti occurrunt, quos singulos pro lubitu vtcunque determinare licet, ita vt istae formulae multo latius pateant, quam eae quas non ita pridem ex iisdem fundamentis derivavi.

Problema 4.

Cum sit, vti jam abunde est demonstratum,

$$\int \frac{x^{\theta-1}-x^{\nu-\theta-1}}{1-x^{\nu}}\,\partial x \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\pi}{\nu \text{ tang. } \frac{\theta\pi}{\nu}},$$

hanc formulam denuo integrare, sumto exponente θ variabili, ita vt integralia evanescant posito $\theta=\alpha$.

Solutio.

§. 11. Quodsi ergo hic per $\partial\theta$ multiplicemus, pro parte dextra habebimus $\frac{\pi\,\partial\theta}{\nu\,\text{tang.}\,\frac{\theta\pi}{\nu}}$, quae formula, posito $\frac{\pi\theta}{\nu}=\Phi$, abit in $\frac{\partial\Phi}{\text{tang.}\,\Phi}=\frac{\partial\Phi\,\text{cos.}\,\Phi}{\text{sin.}\,\Phi}$, cujus integrale manifesto est $l\,\text{sin.}\,\Phi$; quamobrem constanti debita adjecta, pro parte dextra habebimus

$$l\,\text{sin.}\,\frac{\theta\pi}{\nu}-l\,\text{sin.}\,\frac{\alpha\pi}{\nu}=l\,\frac{\text{sin.}\,\frac{\theta\pi}{\nu}}{\text{sin.}\,\frac{\alpha\pi}{\nu}}.$$

Pro parte autem sinistra, quae ita repraesentetur:

$$\int\frac{\partial x}{x}\cdot\frac{x^{\theta}-x^{\nu-\theta}}{1-x^{\nu}},$$

habebimus

$$\int x^{\theta}\,\partial\theta=\frac{x^{\theta}-x^{\alpha}}{l\,x}\quad\text{et}$$

$$\int x^{\nu-\theta}\,\partial\theta=\frac{x^{\nu-\alpha}-x^{\nu-\theta}}{l\,x},$$

quibus valoribus substitutis orietur sequens aequatio integrata:

$$\int\frac{\partial x}{x\,l\,x}\cdot\frac{(x^{\theta}-x^{\alpha}-x^{\nu-\alpha}+x^{\nu-\theta})}{1-x^{\nu}}\begin{bmatrix}\text{ab}\ x=0\\ \text{ad}\ x=1\end{bmatrix}=l\,\frac{\text{sin.}\,\frac{\theta\pi}{\nu}}{\text{sin.}\,\frac{\alpha\pi}{\nu}}$$

vbi iterum tres exponentes indefiniti occurrunt, α, θ, ν.

Corollarium 1.

§. 12. Cum sit, vti jam ante obseruauimus,

$$x^{\nu-\alpha}-x^{\nu-\theta}=x^{\nu-\alpha-\theta}(x^{\theta}-x^{\alpha}),$$

formula nostra commodius ita per factores exprimi poterit:

$$\int\frac{\partial x}{x\,l\,x}\cdot\frac{(x^{\theta}-x^{\alpha})(1-x^{\nu-\alpha-\theta})}{1-x^{\nu}}\begin{bmatrix}\text{ab}\ x=0\\ \text{ad}\ x=1\end{bmatrix}=l\,\frac{\text{sin.}\,\frac{\theta\pi}{\nu}}{\text{sin.}\,\frac{\alpha\pi}{\nu}},$$

vbi

vbi ſi ſumeremus $\nu = \alpha + \theta$, membrum ſiniſtrum evaneſceret, dextrum autem manifeſto quoque evaneſceret.

Corollarium 2.

§. 13. Quodſi autem hic ſumamus $\nu = 2\alpha$, pro dextra foret ſin. $\frac{\alpha\pi}{\nu} = 1$, vnde hoc caſu formula noſtra integralis erit

$$\int \frac{\partial x}{x\, l x} \cdot \frac{(x^\theta - x^\alpha)(1 - x^{\alpha - \theta})}{1 - x^{2\alpha}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \text{ ſin. } \frac{\theta\pi}{2\alpha},$$

quae forma evidenter in hanc contrahitur:

$$\int \frac{x^{\theta - 1} \partial x}{l x} \cdot \frac{(1 - x^{\alpha - \theta})^2}{1 - x^{2\alpha}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \text{ ſin. } \frac{\theta\pi}{2\alpha}.$$

Scholion.

§. 14. Has igitur egregias integrationes deduximus ex formulis integralibus jam pridem erutis, quatenus in iis exponentes indefiniti occurrunt; quod ſi ergo aliae hujusmodi formulae integrales inſuper innoteſcerent, eas ſimili modo tractare liceret; verum hactenus nullae tales formulae ſunt inventae quae ad hunc ſcopum accommodari poſſunt, quam ob cauſam integrationes hic exhibitae ſumma attentione Geometrarum dignae ſunt exiſtimandae.

Additamentum.

§. 15. Cum nuper oſtendiſſem hujus formulae integralis

$$\int \frac{x^{a-1} \partial x}{l x} \cdot \frac{(1 - x^b)(1 - x^c)}{1 - x^n}$$

a termino $x = 0$ ad terminum $x = 1$ extenſae valorem ita exprimi, vt ſit $l \frac{P}{Q}$, exiſtente

P

$$P = \int \frac{x^{a+b-1}\,\partial x}{(1-x^n)^{1-\frac{c}{n}}} \text{ et } Q = \int \frac{x^{a-1}\,\partial x}{(1-x^n)^{1-\frac{c}{n}}}$$

quae integralia denuo ab $x=0$ ad $x=1$ ſunt extendenda: manifeſtum eſt in hac forma generali plerasque integrationes ſupra inuentas contineri; quamobrem cum illis caſibus valores integralium abſolute exprimantur, operae pretium erit iſtam formam generalem ad illos caſus applicare, vt relatio inter binas formulas integrales P et Q inde innoteſcat. Problema quidem primum et ſecundum huc plane non pertinent. Ex problemate igitur tertio et quarto eos perſcrutemur caſus, quos ad formam noſtram generalem reuocare licet.

Euolutio formulae integralis ſupra §. 8. inuentae.

$$\int \frac{\partial x}{x\,l\,x} \cdot \frac{x^{\nu-\alpha} - x^{\alpha}}{1+x^{\nu}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \nu \end{bmatrix} = l \operatorname{cof.} \frac{\alpha\pi}{2\nu}.$$

§. 16. Quoniam hic denominator eſt $1 + x^{\nu}$, vt is ad formam generalem reducatur, multiplicetur fractio ſupra et infra per $1 - x^{\nu}$, et formula iſta integralis hanc induet formam:

$$\int \frac{\partial x}{x\,l\,x} \cdot \frac{(x^{\nu-\alpha} - x^{\alpha})(1-x^{\nu})}{1-x^{2\nu}}.$$

Hic ante omnia diſpiciendum eſt, vter exponentium $\nu - \alpha$ et α ſit major, vnde duos caſus evolvi conveniet, prouti fuerit vel $\nu - \alpha < \alpha$, hoc eſt $\nu < 2\alpha$, vel $\nu - \alpha > \alpha$, hoc eſt $\nu > 2\alpha$.

§. 17. Sit igitur primo $\nu < 2\alpha$, ſeu $\alpha > \frac{1}{2}\nu$, atque formula integralis ita repraeſentari poterit:

$$\int \frac{x^{\nu-\alpha-1}\,\partial x}{l\,x} \cdot \frac{(1-x^{2\alpha-\nu})(1-x^{\nu})}{1-x^{2\nu}}.$$

Hinc jam comparatione cum forma generali inſtituta manifeſto habebimus $a = \nu - \alpha$, $b = 2\alpha - \nu$ et $c = \nu$, denique $n = 2\nu$, ex quibus valoribus formabuntur ſequentes formulae:

P

$$P = \int \frac{x^{\alpha - 1} \partial x}{\sqrt{(1 - x^{2\nu})}} \text{ et } Q = \int \frac{x^{\nu - \alpha - 1} \partial x}{\sqrt{(1 - x^{2\nu})}}.$$

Ponere etiam potuissemus $b = \nu$ et $c = 2\alpha - \nu$, manentibus $a = \nu - \alpha$ et $n = 2\nu$, hincque prodiissent valores

$$P = \int \frac{x^{2\nu - \alpha - 1} \partial x}{(1 - x^{2\nu})^{\frac{3\nu - 2\alpha}{2\nu}}} \text{ et } Q = \int \frac{x^{\nu - \alpha - 1} \partial x}{(1 - x^{2\nu})^{\frac{3\nu - 2\alpha}{2\nu}}},$$

vtrinque autem erit $l \frac{P}{Q} = l \cot. \frac{\alpha\pi}{2\nu}$.

§. 18. Hinc igitur duas nanciscimur integrationes notatu dignissimas. Cum enim sit $\frac{P}{Q} = \cot. \frac{\alpha\pi}{2\nu}$, hae duae integrationes ita se habebunt:

$$\text{I. } \int \frac{x^{\alpha - 1} \partial x}{\sqrt{1 - x^{2\nu}}} : \int \frac{x^{\nu - \alpha - 1} \partial x}{\sqrt{1 - x^{2\nu}}} = \cot. \frac{\alpha\pi}{2\nu};$$

$$\text{II. } \int \frac{x^{2\nu - \alpha - 1} \partial x}{(1 - x^{2\nu})^{\frac{3\nu - 2\alpha}{2\nu}}} : \int \frac{x^{\nu - \alpha - 1} \partial x}{(1 - x^{2\nu})^{\frac{3\nu - 2\alpha}{2\nu}}} = \cot. \frac{\alpha\pi}{2\nu}.$$

§. 19. Sin autem fuerit $\nu > 2\alpha$, ipsa formula generalis mutatis signis ita debet repraesentari:

$$\int \frac{\partial x}{x\, l x} \cdot \frac{(x^{\alpha} - x^{\nu - \alpha})(1 - x^{\nu})}{1 - x^{2\nu}} = l \text{ tang. } \frac{\alpha\pi}{2\nu},$$

cui aequationi nunc induamus hanc formam:

$$\int \frac{x^{\alpha - 1} \partial x}{l x} \cdot \frac{(1 - x^{\nu - 2\alpha})(1 - x^{\nu})}{1 - x^{2\nu}}$$

vnde iam manifesto habemus $a = \alpha$, $b = \nu - 2\alpha$, $c = \nu$, atque $n = 2\nu$, vnde deducuntur isti valores:

$$P = \int \frac{x^{\nu - \alpha - 1} \partial x}{\sqrt{(1 - x^{2\nu})}} \text{ et } Q = \int \frac{x^{\alpha - 1} \partial x}{\sqrt{(1 - x^{2\nu})}}.$$

Sin autem sumamus $c = \nu - 2\alpha$ et $b = \nu$, manente $a = \alpha$ et $n =$

$n = 2\nu$, reperietur

$$P = \int \frac{x^{\alpha+\nu-1}\,\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}} \text{ et } Q = \int \frac{x^{\alpha-1}\,\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}}.$$

§. 20. Cum nunc utrinque fit $l\frac{P}{Q} = l\,\text{tang.}\frac{\alpha\pi}{2\nu}$ ideoque $\frac{P}{Q} = \text{tang.}\frac{\alpha\pi}{2\nu}$, hinc adipifcimur iterum has duas integrationes:

$$\text{III. } \int \frac{x^{\nu-\alpha-1}\,\partial x}{\sqrt{1-x^{2\nu}}} : \int \frac{x^{\alpha-1}\,\partial x}{\sqrt{1-x^{2\nu}}} = \text{tang.}\frac{\alpha\pi}{2\nu},$$

quae quidem conuenit cum priore antecedentium, fiquidem formulae P et Q tantum inter fe permutantur; altera vero integratio eft noua, fcilicet

$$\text{IV. } \int \frac{x^{\alpha+\nu-1}\,\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}} : \int \frac{x^{\alpha-1}\,\partial x}{(1-x^{2\nu})^{\frac{\nu+2\alpha}{2\nu}}} = \text{tang.}\frac{\alpha\pi}{2\nu}.$$

Euolutio formulae integralis §. 9. allatae:

$$\int \frac{x^{\theta-1}\,\partial x}{l\,x} \cdot \frac{1-x^{2\alpha-2\theta}}{1+x^{2\alpha}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l\,\text{tang.}\frac{\theta\pi}{4\alpha}.$$

§. 21. Quo haec expreffio ad formam praefcriptam reducatur, multiplicetur fupra et infra per $1-x^{2\alpha}$, vt habeamus hanc formam:

$$\int \frac{x^{\theta-1}\,\partial x}{l\,x} \cdot \frac{(1-x^{2\alpha-2\theta})(1-x^{2\alpha})}{1-x^{4\alpha}} = l\,\text{tang.}\frac{\theta\pi}{4\alpha},$$

quae fponte ad formam generalem reuocatur, fumendo $a = \theta$, $b = 2\alpha - 2\theta$, $c = 2\alpha$ et $n = 4\alpha$, fi modo fuerit $\alpha > \theta$. Si enim fuerit $\theta > \alpha$, alio modo comparatio inftitui debet, vti deinceps videbimus. Ex his autem valoribus conficietur

$$P = \int \frac{x^{2\alpha-\theta-1}\,\partial x}{\sqrt{(1-x^{4\alpha})}} \text{ et } Q = \int \frac{x^{\theta-1}\,\partial x}{\sqrt{(1-x^{4\alpha})}},$$

vnde

vnde ergo deducitur

$$\text{V.}\ \int \frac{x^{2\alpha-\theta-1}\,\partial x}{\sqrt{(1-x^{4\alpha})}} : \int \frac{x^{\theta-1}\,\partial x}{\sqrt{(1-x^{4\alpha})}} = \text{tang.}\,\frac{\theta\pi}{4\alpha}.$$

§. 22. Possumus etiam valores litterarum b et c inter se permutare, vt sit $b = 2\alpha$ et $c = 2\alpha - 2\theta$, manentibus $a = \theta$ et $n = 4\alpha$; tum autem fiet

$$P = \int \frac{x^{2\alpha+\theta-1}\,\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}} \text{ et } Q = \int \frac{x^{\theta-1}\,\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}},$$

hincque deducitur reductio

$$\text{VI.}\ \int \frac{x^{2\alpha+\theta-1}\,\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}} : \int \frac{x^{\theta-1}\,\partial x}{(1-x^{4\alpha})^{\frac{\alpha+\theta}{2\alpha}}} = \text{tang.}\,\frac{\theta\pi}{4\alpha},$$

quae autem, aeque ac praecedens, locum non habet, nisi sit $\alpha > \theta$.

§. 23. Quod si autem θ superet α, aequationem nostram in aliam formam transfundi oportet, signa vtrinque mutando, vnde prodibit

$$\int \frac{x^{2\alpha-\theta-1}\,\partial x}{l\,x} \cdot \frac{(1-x^{2\theta-2\alpha})(1-x^{2\alpha})}{1-x^{4\alpha}} = l\cot.\frac{\theta\pi}{4\alpha}.$$

Hic iam iterum duplex comparatio institui potest: primo scilicet sumamus $a = 2\alpha - \theta$, $b = 2\theta - 2\alpha$, $c = 2\alpha$ et $n = 4\alpha$, vnde formamus

$$P = \int \frac{x^{\theta-1}\,\partial x}{\sqrt{(1-x^{4\alpha})}} \text{ et } Q = \int \frac{x^{2\alpha-\theta-1}\,\partial x}{\sqrt{(1-x^{4\alpha})}}$$

hincque oritur septima relatio haec:

$$\text{VII.}\ \int \frac{x^{\theta-1}\,\partial x}{\sqrt{(1-x^{4\alpha})}} : \int \frac{x^{2\alpha-\theta-1}\,\partial x}{\sqrt{(1-x^{4\alpha})}} = \cot.\frac{\theta\pi}{4\alpha},$$

quae manifesto cum quinta congruit.

§. 24. Noua autem reductio obtinebitur, si statuamus $b = 2\alpha$ et $c = 2\theta - 2\alpha$, manentibus $a = 2\alpha - \theta$ et $n = 4\alpha$; tum igitur erit

$$P = \int \frac{x^{4\alpha - \theta - 1}\,\partial x}{(1 - x^{4\alpha})^{\frac{3\alpha - \theta}{2\alpha}}} \text{ et } Q = \int \frac{x^{2\alpha - \theta - 1}\,\partial x}{(1 - x^{4\alpha})^{\frac{3\alpha - \theta}{2\alpha}}}.$$

Hinc vero colligitur reductio octaua

$$\text{VIII.} \int \frac{x^{4\alpha - \theta - 1}\,\partial x}{(1 - x^{4\alpha})^{\frac{3\alpha - \theta}{2\alpha}}} : \int \frac{x^{2\alpha - \theta - 1}\,\partial x}{(1 - x^{4\alpha})^{\frac{3\alpha - \theta}{2\alpha}}} = \cot.\frac{\theta\pi}{4\alpha}.$$

§. 25. Hic autem probe notandum est, quaternas posteriores reductiones ex quatuor prioribus oriri, si in istis loco α scribatur θ, at 2α loco ν, ita vt quatuor posteriores reductiones iam in prioribus contineantur; quamobrem siue quatuor priores, siue posteriores, penitus omittere licebit, ita vt nobis tantum quatuor relinquantur, inter quas porro, quoniam tertia non discrepat a prima, tantum tres supererunt huiusmodi reductiones, quae quidem ex problemate tertio sunt natae.

Euolutio formulae integralis §. 12. allatae:

$$\int \frac{\partial x}{x\,l\,x} \cdot \frac{(x^{\theta} - x^{\alpha})(1 - x^{\nu - \alpha - \theta})}{1 - x^{\nu}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l\,\frac{\sin.\frac{\theta\pi}{\nu}}{\sin.\frac{\alpha\pi}{\nu}}.$$

§. 26. Ista expressio iam congruit cum forma nostra generali, neque idcirco vlteriori transformatione indiget. Hic quidem duo casus essent distinguendi, prouti fuerit vel $\theta > \alpha$, vel $\theta < \alpha$; verum hac etiam distinctione carere possumus, propterea quod binae litterae α et θ inter se sunt permutabiles: iis enim permutatis signa vtrinque inuertuntur. Hanc ob caussam, quoscunque valores habuerint ambae litterae α et θ, minorem semper littera θ, maiorem vero littera α designare licebit, vnde aequatio nostra ita repraesentabitur:

$\int$

$$\int \frac{x^{\theta-1}\,\partial x}{l\,x} \cdot \frac{(1-x^{\alpha-\theta})(1-x^{\nu-\alpha-\theta})}{1-x^{\nu}} = l\,\frac{\text{fin.}\frac{\theta\pi}{\nu}}{\text{fin.}\frac{\alpha\pi}{\nu}}.$$

§. 27. Nihilo vero minus duo cafus diftinguendi etiam hic occurrunt, prouti fuerit vel $\nu > \alpha + \theta$, vel $\nu < \alpha + \theta$. Sit igitur primo $\nu > \alpha + \theta$, et forma expofita manebit inuariata, quae denuo duplicem comparationem cum generali admittit. Primo igitur ftatuamus $a = \theta$, $b = \alpha - \theta$, $c = \nu - \alpha - \theta$ et $n = \nu$, qui valores nobis fuppeditant

$$P = \int \frac{x^{\alpha-1}\,\partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}} \text{ et } Q = \int \frac{x^{\theta-1}\,\partial x}{(1-x^{\nu})^{\frac{\alpha-\theta}{\nu}}}$$

ficque ex hac euolutione habebimus fequentem reductionem:

$$\text{I. } \int \frac{x^{\alpha-1}\,\partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}} : \int \frac{x^{\theta-1}\,\partial x}{(1-x^{\nu})^{\frac{\alpha+\theta}{\nu}}} = \frac{\text{fin.}\frac{\theta\pi}{\nu}}{\text{fin.}\frac{\alpha\pi}{\nu}}.$$

§. 28. Secunda nafcetur reductio permutandis litteris b et c, ita vt fit $a = \theta$, $b = \nu - \alpha - \theta$, $c = \alpha - \theta$, et $n = \nu$, vnde formantur hae formulae:

$$P = \int \frac{x^{\nu-\alpha-1}\,\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} \text{ et } Q = \int \frac{x^{\theta-1}\,\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}}$$

quare fecunda reductio hinc orta erit

$$\text{II. } \int \frac{x^{\nu-\alpha-1}\,\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} : \int \frac{x^{\theta-1}\,\partial x}{(1-x^{\nu})^{\frac{\nu-\alpha+\theta}{\nu}}} = \frac{\text{fin.}\frac{\theta\pi}{\nu}}{\text{fin.}\frac{\alpha\pi}{\nu}},$$

quae duae reductiones poftulant vt fit $\nu > \alpha + \theta$.

§. 29. Sin autem fuerit $\nu < \alpha + \theta$, ipfa aequationis forma hoc modo immutari debebit:

$$\int \frac{x^{\nu-\alpha-1}\,\partial x}{l\,x} \cdot \frac{(1-x^{\alpha-\theta})(1-x^{\alpha+\theta-\nu})}{1-x^{\nu}} = l\,\frac{\text{fin.}\frac{\alpha\pi}{\nu}}{\text{fin.}\frac{\theta\pi}{\nu}}$$

 vbi

vbi iterum gemina comparatio institui potest. Sit igitur primo $a = \nu - \alpha$, $b = \alpha - \theta$, $c = \alpha + \theta - \nu$ et $n = \nu$, vnde oriuntur hae formulae:

$$P = \int \frac{x^{\nu - \theta - 1}\,\partial x}{(1 - x^\nu)^{\frac{2\nu - \alpha - \theta}{\nu}}} \text{ et } Q = \int \frac{x^{\nu - \alpha - 1}\,\partial x}{(1 - x^\nu)^{\frac{2\nu - \alpha - \theta}{\nu}}}.$$

Hinc igitur concluditur tertia reductio:

$$\text{III. } \int \frac{x^{\nu - \theta - 1}\,\partial x}{(1 - x^\nu)^{\frac{2\nu - \alpha - \theta}{\nu}}} : \int \frac{x^{\nu - \alpha - 1}\,\partial x}{(1 - x^\nu)^{\frac{2\nu - \alpha - \theta}{\nu}}} = \frac{\text{fin. } \frac{\alpha\pi}{\nu}}{\text{fin. } \frac{\theta\pi}{\nu}}.$$

§. 30. Denique statuamus $a = \nu - \alpha$, $b = \alpha + \theta - \nu$, $c = \alpha - \theta$ et $n = \nu$, et formulae hinc sequentes nascentur:

$$P = \int \frac{x^{\theta - 1}\,\partial x}{(1 - x^\nu)^{\frac{\nu - \alpha + \theta}{\nu}}} \text{ et } Q = \int \frac{x^{\nu - \alpha - 1}\,\partial x}{(1 - x^\nu)^{\frac{\nu - \alpha + \theta}{\nu}}}$$

ita vt quarta hinc oriatur reductio:

$$\text{IV. } \int \frac{x^{\theta - 1}\,\partial x}{(1 - x^\nu)^{\frac{\nu - \alpha + \theta}{\nu}}} : \int \frac{x^{\nu - \alpha - 1}\,\partial x}{(1 - x^\nu)^{\frac{\nu - \alpha + \theta}{\nu}}} = \frac{\text{fin. } \frac{\alpha\pi}{\nu}}{\text{fin. } \frac{\theta\pi}{\nu}}.$$

§. 31. Quatuor igitur hic nacti sumus formularum integralium paria, quae eandem inter se tenent rationem ac sinus duorum angulorum; dum euolutiones praecedentes tantum tria huiusmodi paria praebuerant, quarum ratio P:Q tangenti cuiuspiam anguli aequatur, vbi quidem euidens est secundam et quartam inter se conuenire. Cum igitur huiusmodi reductiones altioris sint indaginis, ac sine dubio insignem vsum habere queant, opere pretium erit eas clarius ob oculos exponere.

Problema.

§. 32. *Inuenire binas formulas integrales P et Q ab* $x = 0$ *ad* $x = 1$ *extensas, vt fiat* $\frac{P}{Q} = \text{tang. } \frac{m\pi}{2n}$.

So-

Solutio.

Triplici igitur modo hoc fieri poteſt, ſecundum euolutionem primam ſupra inſtitutam. I. Ex prima enim reductione, cum ſit cot. $\frac{\alpha\pi}{2\nu} = $ tang. $\frac{(\nu-\alpha)\pi}{2\nu}$, fiet $\nu - \alpha = m$ et $\nu = n$, ita vt ſit $\alpha = n - m$. Hinc igitur erit

$$P = \int \frac{x^{n-m-1}\,\partial x}{\sqrt{(1-x^{2n})}} \text{ et } Q = \int \frac{x^{m-1}\,\partial x}{\sqrt{(1-x^{2n})}},$$

quae ergo eſt ſolutio prima. II. Secunda reductio ſupra allata erat $\frac{P}{Q} = $ cot. $\frac{\alpha\pi}{2\nu} = $ tang. $\frac{(\nu-\alpha)\pi}{2\nu}$, vbi ergo iterum eſt $\alpha = n - m$ et $\nu = n$, ſicque ſecunda ſolutio huius problematis conſtabit his formulis:

$$P = \int \frac{x^{m+n-1}\,\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}} \text{ et } Q = \int \frac{x^{m-1}\,\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}}.$$

Hae autem formulae tantum valent, quando fuerit $m < \frac{1}{2}a$, ideoque ipſe angulus $\frac{m\pi}{2n}$ minor ſemirecto. III. Quoniam tertia reductio ibi allata cum prima conuenit, ex quarta, vbi erat $\frac{P}{Q} = $ tang. $\frac{\alpha\pi}{2\nu}$, ideoque pro noſtro caſu $\alpha = m$ et $\nu = n$, tertia ſolutio ita ſe habebit:

$$P = \int \frac{x^{m+n-1}\,\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}} \text{ et } Q = \int \frac{x^{m-1}\,\partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}},$$

qui valores quoniam a praecedentibus non ſunt diuerſi, duas tantum adipiſcimur ſolutiones noſtri problematis, quarum ſecunda limitatione quadam indiget, ſcilicet $m < \frac{1}{2}n$, prior vero ad omnes angulos recto non maiores patet. Hae ergo duae ſolutiones ita repraeſententur:

$$\text{I. } P = \int \frac{x^{n-m-1}\,\partial x}{\sqrt{(1-x^{2n})}},\quad Q = \int \frac{x^{m-1}\,\partial x}{\sqrt{(1-x^{2n})}},$$

$$\text{II. } P = \int \frac{x^{m+n-1} \partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}}, \quad Q = \int \frac{x^{m-1} \partial x}{(1-x^{2n})^{\frac{2m+n}{2n}}};$$

ex vtraque igitur erit $\frac{P}{Q} = \text{tang.} \frac{m\pi}{2n}$.

Problema.

§. 33. *Inuenire binas formulas integrales* P *et* Q, *vt fiat* $\frac{P}{Q} = \frac{\text{fin.} \frac{p\pi}{2n}}{\text{fin.} \frac{q\pi}{2n}}$, *fiquidem ambo illa integralia ab* $x=0$ *ad* $x=1$ *extendantur.*

Solutio.

Ad hanc igitur formam transferamus quatuor illas reductiones in euolutione tertia traditas, et cum pro prima et fecunda effet $\frac{P}{Q} = \frac{\text{fin.} \frac{\theta\pi}{\nu}}{\text{fin.} \frac{\alpha\pi}{\nu}}$, pro forma hic praefcripta erit $\theta = p$, $\alpha = q$ et $\nu = 2n$, quamobrem hinc nancifcimur duas fequentes folutiones:

$$\text{I. } P = \int \frac{x^{q-1} \partial x}{(1-x^{2n})^{\frac{p+q}{2n}}} \text{ et } Q = \int \frac{x^{p-1} \partial x}{(1-x^{2n})^{\frac{p+q}{2n}}},$$

$$\text{II. } P = \int \frac{x^{2n-q-1} \partial x}{(1-x^{2n})^{\frac{2n-q+p}{2n}}} \text{ et } Q = \int \frac{x^{p-1} \partial x}{(1-x^{2n})^{\frac{2n-q+p}{2n}}}.$$

Tertia vero et quarta reductio habebant $\frac{P}{Q} = \frac{\text{fin.} \frac{\alpha\pi}{\nu}}{\text{fin.} \frac{\theta\pi}{\nu}}$, pro qua igitur erit $\alpha = p$, $\theta = q$, $\nu = 2n$, vnde ambae folutiones fequentes deducuntur:

$$\text{III. } P = \int \frac{x^{2n-q-1} \partial x}{(1-x^{2n})^{\frac{4n-p-q}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1} \partial x}{(1-x^{2n})^{\frac{4n-p-q}{2n}}},$$

IV.

$$\text{IV. } P = \int \frac{x^{q-1}\partial x}{(1-x^{2n})^{\frac{2n-p+q}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{2n-p+q}{2n}}}.$$

Hinc igitur patet quadruplici modo fieri posse $\frac{P}{Q} = \frac{\text{fin.}\frac{p\pi}{2n}}{\text{fin.}\frac{q\pi}{2n}}$.

Corollarium 1.

§. 34. Si assumamus $q = n$, vt fiat $\text{fin.}\frac{q\pi}{2n} = 1$, ideoque prodire debeat $\frac{P}{Q} = \text{fin.}\frac{p\pi}{2n}$; pro hoc casu quatuor inuentae solutiones dabunt

$$\text{I. } P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{p+n}{2n}}} \text{ et } Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{p+n}{2n}}},$$

$$\text{II. } P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{n+p}{2n}}} \text{ et } Q = \int \frac{x^{p-1}\partial x}{(1-x^{2n})^{\frac{n+p}{2n}}},$$

$$\text{III. } P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}},$$

$$\text{IV. } P = \int \frac{x^{n-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1}\partial x}{(1-x^{2n})^{\frac{3n-p}{2n}}},$$

vbi ergo solutio prima cum secunda et tertia cum quarta conuenit.

Corollarium 2.

§. 35. Sumamus nunc esse $q = n - p$, vt fiat $\text{fin.}\frac{q\pi}{2n} = \text{cof.}\frac{p\pi}{2n}$, ideoque prodire debeat $\frac{P}{Q} = \text{tang.}\frac{p\pi}{2n}$. Pro hoc ergo casu quatuor solutiones inuentae euadent

$$\text{I. } P = \int \frac{x^{n-p-1}\partial x}{\sqrt{(1-x^{2n})}} \text{ et } Q = \int \frac{x^{p-1}\partial x}{\sqrt{(1-x^{2n})}},$$

$$\text{II. } P = \int \frac{x^{n+p-1}\,\partial x}{(1-x^{2n})^{\frac{n+2p}{2n}}} \text{ et } Q = \int \frac{x^{p-1}\,\partial x}{(1-x^{2n})^{\frac{n+2p}{2n}}};$$

$$\text{III. } P = \int \frac{x^{n+p-1}\,\partial x}{(1-x^{2n})^{\frac{3}{2}}} \text{ et } Q = \int \frac{x^{2n-p-1}\,\partial x}{(1-x^{2n})^{\frac{3}{2}}},$$

$$\text{IV. } P = \int \frac{x^{n-p-1}\,\partial x}{(1-x^{2n})^{\frac{3n-2p}{2n}}} \text{ et } Q = \int \frac{x^{2n-p-1}\,\partial x}{(1-x^{2n})^{\frac{3n-2p}{2n}}},$$

hincque erit $\frac{P}{Q} = \text{tang.}\, \frac{p\pi}{2n}$,

vbi prima et fecunda forma cum iis quas in praecedente problemate inuenimus prorfus conueniunt; tertia autem forma, ob $(1-x^{2n})^{\frac{3}{2}}$, fit incongrua, quia inde P et Q in infinitum excrefcerent; quarta autem nouam formam dare videtur.

METHO-